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Note

A Note on Rational and Spline Approximation

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In a recent technically involved but important paper (in my opinion), P. Petrushev [2] established a weak-type estimate of the rational approximation by the best spline approximation. This was done following the classical result of D. J. Newman [1] about approximation of the function $|x|$ by rational functions. In the present note, we make some observations that yield a substantial improvement in the main result of [2]. The reader should note that this improvement also eluded Petrushev and Popov in their recent text on rational approximation [3, Chap. 8].

We define $R_n(f)_p$, the rate of rational approximation, by

$$R_n(f)_p = \inf\{\|f - R\|_{L_p(-\infty,\infty)} : R \in \mathscr{R}_n\}, \tag{1}$$

where $\mathscr{R}_n$ is the class of rational functions of degree n, i.e., quotients of the type P_n/Q_n, where both P_n and Q_n are polynomials of degree n. We define $S_m^k(f)_p$, the rate of spline approximation, by

$$S_m^k(f)_p = \inf\{\|f - \varphi\|_{L_p(-\infty,\infty)} : \varphi \in S(k; m)\}, \tag{2}$$

where $S(k; m)$ is the set of spline functions with $m+1$ arbitrary knots x_i, $-\infty < x_0 < \cdots < x_m < \infty$; that is, $\varphi \in S(k; m)$ is a polynomial of degree $k-1$ in (x_i, x_{i+1}), $(-\infty, x_0)$, and (x_m, ∞). We observe that if φ is a polynomial (of any degree) in $(-\infty, x_0)$ or (x_m, ∞) and $f \in L_p(R)$, φ will be zero in those intervals. We further note that the question whether we have $m+1$ free knots or $m+3$ (including $\pm\infty$) is immaterial as the results discussed here are asymptotic.

One can prove the following result.

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640/58/3-8

THEOREM 1. *If* $f \in L_p(-\infty, \infty)$, $1 \leqslant p < \infty$, $k \geqslant 1$, *and* $\alpha > 0$, *then for* $n \geqslant n_0$,

$$R_n(f)_p \leqslant C n^{-\alpha} \sum_{v=1}^{n} v^{\alpha-1} S_v^k(f)_p, \tag{3}$$

where $C = C(p, k, \alpha)$.

We note that, in Petrushev's paper, in the analogous result (Theorem 2.1, called there the main result) f is required to have $[a, b]$ as support, in which case one has $S_v^k(f)_p = S_v^k(f, [a, b])$.

Proof. The main tool in the proof of Theorem 2.1 in [2] is Theorem 3.1 there, the proof of which occupies most of that paper. We observe that in the statement and proof of Theorem 3.1, the constant D of the estimate does not depend on the given interval $[a, b]$. Therefore, we may use Theorem 3.1 for different compact intervals at each step of the proof. Following Petrushev's proof, we may choose $\varphi_m \in S(k; m)$ such that, for the given $f (f \in L_p(-\infty, \infty))$,

$$\|f - \varphi_m\|_{L_p(-\infty, \infty)} \leqslant 2S_m^k(f)_p. \tag{4}$$

We observe that the support of φ_m is a finite interval, say, $[a_m, b_m]$. We further follow Petrushev to observe that $\varphi_{2^i} - \varphi_{2^{i-1}} \in S(k, 2^{i+1})$ and its support is in $[\min(a_{2^i}, a_{2^{i-1}}), \max(b_{2^i}, b_{2^{i-1}})]$.

Petrushev's result can now be applied to $\varphi_{2^i} - \varphi_{2^{i-1}}$ and the rest of the proof follows Petrushev's proof verbatim.

REFERENCES

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